Functions:

D: $f: D \subset \mathbb{R}^2 \to \mathbb{R}$ has limit L as $(x,y) \to (x_0,y_0)$

$\iff \lim_{x,y \to x_0,y_0} f(x,y) = L$

$\iff$ When $(x,y) \to (x_0,y_0)$ along ANY path in D, $f(x,y)$ gets close to L.

Make rigorous with $\varepsilon - N$

Use to show limit does not exist

- The limit can exist even when f is undefined at the point
- L must be finite

T: $c, L, M \in \mathbb{R}$, $f \to L$, $g \to M$
- $f+g \to L+M$
- $cf \to cL$
- $fg \to LM$
- $f/g \to L/M$ when $M \neq 0$.

use to find limit

T: f, g, h continuous & $g(x,y) \leq f(x,y) \leq h(x,y)$

$\implies [\, g \to L \;\&\; h \to L \implies f \to L \,]$

Use to prove limit is something in general.

D: f continuous at (x_0,y_0)

$\iff \lim_{x,y \to x_0,y_0} f(x,y) = f(x_0,y_0)$

T: polynomials, trig, exponentials, n^{th} roots, logs & hyperbolic functions are all continuous on their domains.

T: f & g cont at (x_0,y_0), $c \in \mathbb{R}$

Then the following are cont there
- $f+g$
- cf
- fg
- f/g $(g \neq 0)$
- $h \circ f$ with h cont at $f(x_0,y_0)$.

D: f is differentiable at (x_0,y_0)

$\iff f_x$ & f_y exist at (x_0,y_0) — Partial derivatives (Defined as $\mathbb{R}$ derivatives with Lim)

& The tangent plane

$$z = f(x,y) = f(x_0,y_0) + f_x\Big|_{(x_0,y_0)}(x-x_0) + f_y\Big|_{(x_0,y_0)}(y-y_0)$$

is a good approximation to f at (x_0,y_0)

↳ In the sense that can be made rigorous through the higher order terms $\to 0$ in the limit.

T: f_x & f_y exist & are continuous at (x_0,y_0) $\implies$ f is differentiable at (x_0,y_0)

Note strictly a one way implication.

D: f is C^n if all of its n^{th} order partial derivatives exist & are continuous.

T: f is $C^n \Rightarrow f$ is $C^1, \dots, C^{n-1}$

T: f $C^1 \implies f$ differentiable.

T: Chain Rule: Can generalize to $\mathbb{R}^n$

$h(x,y,z) = f(u(x,y,z), v(x,y,z), w(x,y,z))$

Then

$h_x = f_u u_x + f_v v_x + f_w w_x$

h_y ...

h_z ...

$$[h_x \; h_y \; h_z] = [f_u \; f_v \; f_w] \underbrace{\begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix}}$$

D: Jacobian $= \dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \det(\;)$

T: Jacobian $\neq 0$

$$\implies \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix} \begin{bmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{bmatrix} = I_3$$

D: For $f:\mathbb{R}^n \to \mathbb{R}^m$ differentiable then the derivative is

$$D_f = \begin{array}{c} \xrightarrow{x_i} \\ \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix} \end{array} \downarrow f_i$$

where $f = (f_1, \dots, f_m)$

$m \times n$ matrix.

T: $f:\mathbb{R}^m \to \mathbb{R}^p$, $g:\mathbb{R}^n \to \mathbb{R}^m$ differentiable. & $f\circ g$ is defined

$$\Longrightarrow D_{f\circ g} = D_f D_g$$

Derivative of composition is product of matrices

D: For $\mathbb{R}^n$ with basis $\{e_1, \dots, e_n\}$ then define $\nabla = \sum_{i=1}^{n} e_i \frac{\partial}{\partial x_i} = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$

T: If f is C^n we can approximate it by a polynomial order n around a point a by

$$P_n(x) = \sum_{k=0}^{n} \frac{1}{k!}\left[(x-a)\cdot\nabla\right]^k f\Big|_a$$

Dot product.

D: The truncation error $R_n(x)$ about point a is

$$R_n(x) = f(x) - P_n(x)$$

T: $R_n(x) = \frac{1}{(n+1)!}\left[(x-a)\cdot\nabla\right]^{n+1} f\Big|_{(a+\xi(x-a))}$

For $\xi \in (0,1)$

D: The critical points of function f occur when $\nabla f = 0$ or does not exist.

D: For f a func of 2 variables the Hessian matrix is

$$H(a,b) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}\Bigg|_{(a,b)}$$

D: The hessian is

$$\det(H) = f_{xx}f_{yy} - (f_{xy})^2$$

T: Hessian determines type of critical point. So for a critical point (a,b)

$\det(H(a,b)) = 0 \Rightarrow$ Inconclusive Typically try higher degree taylor approx

$\det(H(a,b)) < 0 \Rightarrow$ Saddle point at (a,b)

$\det(H(a,b)) > 0 \Rightarrow [f_{xx}(a,b) < 0 \Rightarrow \max \text{ at } (a,b)]$

OR $[f_{xx}(a,b) > 0 \Rightarrow \min \text{ at } (a,b)]$

T: f & g differentiable, $\underline{a}$ an extrema of f subject to the constraint that $g(\underline{x}) = 0$

$$\Longrightarrow \exists \lambda \in \mathbb{R} \quad \nabla f(\underline{a}) = \lambda \nabla g(\underline{a})$$

OR. $\nabla g(\underline{a}) = 0$

D: This λ is called a Lagrange multiplier.

D: $D \subseteq \mathbb{R}$ bounded $\iff (\exists M \in \mathbb{R})(\forall x \in D)(|x| < M)$

D: $D \subseteq \mathbb{R}$ is closed if it contains all its boundary points.

T: $\{x \in \mathbb{R}^n \mid g(x) = 0\}$ closed & bounded $\Longrightarrow \exists$ a max & min of f subject to $g(x) = 0$.

T: f, g_1, g_2 differentiable. $\underline{a}$ an extrema of f subject to $g_1(x) = 0$, $g_2(x) = 0$.

$\Longrightarrow \exists \lambda_1, \lambda_2 \in \mathbb{R}$

$$\nabla f(\underline{a}) = \lambda_1 \nabla g_1(\underline{a}) + \lambda_2 \nabla g_2(\underline{a})$$

Assuming that $\nabla g_1(\underline{a})$ & $\nabla g_2(\underline{a})$ are linearly independent.

Space Curves & Vector Fields:

We start by considering a $c(t)=(x(t),y(t),z(t))$ to be a parametrisation of some curve C, describing the path of a particle at time t.

c gives the location of a particle on curve C at time t.

D: Velocity: $v(t)=\frac{dc}{dt}=\left(\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right)$

D: Speed $=|v(t)|=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}$

The direction of v is tangent to the path.

D: acceleration: $a(t)=\frac{d^2c}{dt^2}=\left(\frac{d^2x}{dt^2},\frac{d^2y}{dt^2},\frac{d^2z}{dt^2}\right)$

D: Tangent line to c at $t=t_0$:

$$\ell(t)=c(t_0)+(t-t_0)c'(t_0)$$

Recall:

$\underline{a},\underline{b}\in\mathbb{R}^n \Rightarrow \underline{a}\cdot\underline{b}=\sum_{i=1}^{n}a_ib_i$

① $\underline{a}\cdot\underline{b}=\underline{b}\cdot\underline{a}$ ② $\underline{a}\cdot\underline{b}=|\underline{a}||\underline{b}|\cos(\theta)$

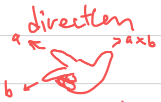
③ $\underline{a}\cdot(\underline{b}+\underline{c})=\underline{a}\cdot\underline{b}+\underline{a}\cdot\underline{c}$

④ $\lambda\in\mathbb{R}$ $\lambda(\underline{a}\cdot\underline{b})=(\lambda\underline{a})\cdot\underline{b}=\underline{a}\cdot(\lambda\underline{b})$

⑤ $\underline{a},\underline{b}\neq 0 \Rightarrow [\underline{a}\perp\underline{b} \Leftrightarrow a\cdot b=0]$

$\underline{a},\underline{b}\in\mathbb{R}^3 \Rightarrow \underline{a}\times\underline{b}=|\underline{a}||\underline{b}|\sin(\theta)\underline{n}$

- θ angle between a & b
- $\underline{n}$ is unit vector $\perp$ to plane defined by $\underline{a}$ & $\underline{b}$ in direction of right hand rule



$\underline{a}\times\underline{b}$ is a vector perpendicular to both $\underline{a}$ & $\underline{b}$.

$$\underline{a}\times\underline{b}=(a_2b_3-a_3b_2,\ a_3b_1-a_1b_3,\ a_1b_2-a_2b_1)$$

$$=\det\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$i, j, k \rightarrow$ basis vectors.

① $\underline{a}\times\underline{a}=0$ ② $\underline{a}\times\underline{b}=-(\underline{b}\times\underline{a})$

③ $\underline{a}\times(\underline{b}+\underline{c})=(\underline{a}\times\underline{b})+(\underline{a}\times\underline{c})$

④ $\lambda\in\mathbb{R}$ $(r\underline{a})\times\underline{b}=\underline{a}\times(r\underline{b})=r(\underline{a}\times\underline{b})$

T: b & c differentiable $\mathbb{R}^3$ paths

① $\frac{d}{dt}[b+c]=\frac{db}{dt}+\frac{dc}{dt}$

② $\frac{d}{dt}[b\cdot c]=\frac{db}{dt}\cdot c+b\cdot\frac{dc}{dt}$

③ $\frac{d}{dt}[b\times c]=\frac{db}{dt}\times c+b\times\frac{dc}{dt}$

D: The **length or arclength** of a path c

$$s=\int_a^b|c'(t)|\,dt$$

D: Unit tangent to a curve c at $c(t)$

$$T(t)=\frac{dc}{ds}=\left(\frac{dc}{dt}\right)\Big/\left|\frac{dc}{dt}\right|$$

D: Principal normal

$$N(t)=\frac{\left(\frac{dT}{dt}\right)}{\left|\frac{dT}{dt}\right|}$$

D: Binormal vector

$$B(t)=T(t)\times N(t)$$

D: Curvature at $c(t)$ on C

$$\kappa(t)=\left|\frac{dT}{ds}\right|=\left|\frac{dT}{dt}\right|\Big/\left|\frac{ds}{dt}\right|$$

D: Torsion $\tau(t)$

$$\frac{dB}{ds}=\frac{B'(t)}{|c'(t)|}=-\tau(t)N(t)$$

D: A **vector field** is a function $F:\mathbb{R}^n\longrightarrow\mathbb{R}^n$

D: A path c is a **flow line** (or stream line) of a vector field F if $c'(t)=F(c(t))$.

D: In $\mathbb{R}^3$ $\nabla=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)$

Note if f is C^1 ∇f is an $\mathbb{R}^3$ vector field.

D: For a vector field $F=(F_1,F_2,F_3)$ C^1 the **divergence** of F is a scalar function: $\mathrm{div}(F)=\nabla\cdot F$

$$=\frac{\partial}{\partial x}F_1+\frac{\partial}{\partial y}F_2+\frac{\partial}{\partial z}F_3$$

Note $\nabla\cdot F>0 \Rightarrow$ source at P

$\nabla\cdot F<0 \Rightarrow$ Sink at P

$\nabla\cdot F=0 \Rightarrow$ Incompressable vector field.

D: Curl of F is itself a vector field

$$\mathrm{Curl}(F)=\nabla\times F=\det\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

D: The Laplacian operator is $\nabla^2 = \nabla \cdot \nabla$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Maps C^2 scalar f to

$$\nabla^2 f = f_{xx} + f_{yy} + f_{zz}.$$

If F is a vector field

$$\nabla^2 F = (\nabla^2 F_1, \nabla^2 F_2, \nabla^2 F_3)$$

T: V a C^1 vector field

$\nabla \times V = 0 \implies \exists \phi$ a scalar function such that $V = \nabla \phi$

ϕ is unique up to an unknown constant

V is called the gradient field & ϕ the scalar potential.

T: V C^1 vector field

$\nabla \cdot V = 0 \implies \exists F$ a vector field with $V = \nabla \times F$

F is not unique.

F here is the vector potential.

Integrals:

(For $\mathbb{R}^2$)

T: Fubini's Thm: Order of integration can be changed if
- The domain can be divided into horrozontal & verticle strips

AND • f is cts in the domain.

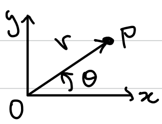
T: f cts over a rectangular solid (ie $B=[a,b]\times[c,d]\times[p,q]$) then the order of integration can be changed.

D: A domain D is an elementary region in $\mathbb{R}^3$ if one variable is bounded by functions of 2-variables and the domain of these functions can be devided into both horrozontal & verticle strips.

Coordinate Systems:

2 DIM – Cartesian (x,y).

Polar : (r,θ)



$r=|\vec{OP}| = \sqrt{x^2+y^2}$

$x=r\cos(\theta)$, $y=r\sin(\theta)$

Note that θ is not unique when defined as the angle counter clockwise from the positive x axis.
After restricting to $\theta\in[0,2\pi)$

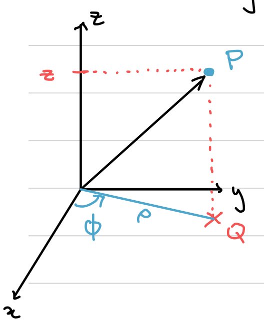
$$\theta(x,y)=\begin{cases}\arctan\left(\frac{y}{x}\right), & x>0,\ y\geq 0\\ \pi+\arctan\left(\frac{y}{x}\right), & x<0\\ 2\pi+\arctan(y/x), & x>0,\ y<0\\ \frac{\pi}{2}, & x=0,\ y>0\\ 3\pi/2, & x=0,\ y<0\end{cases}$$

For $\arctan\left(\frac{y}{x}\right)\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

$\hat{r} = \frac{r}{|r|} = (\cos(\theta),\sin(\theta))$

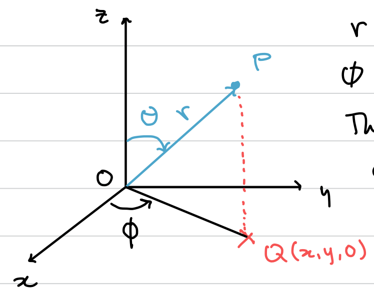
$\hat{\theta} = \frac{1}{|r|}(-y,x) = (-\sin(\theta),\cos(\theta))$

3 DIM – Cylindrical Coordinates (ρ,ϕ,z)



Project P into x-y plane giving point $Q=(x,y,0)$
Then take the polar coordinates of Q, r,θ
then $\rho=r$, $\phi=\theta$.
z is just the z coordinate.

Spherical Coordinates (r,θ,ϕ)



$r=|\vec{OP}|=\sqrt{x^2+y^2+z^2}$
ϕ is the azimuthal angle
The same ϕ as in cyclindrical coordinates.

θ is the polar angle . $\theta\in[0,\pi]$
$\theta=\arccos(z/r)$

$x=r\sin(\theta)\cos(\phi)$
$y=r\sin(\theta)\sin(\phi)$
$z=r\cos(\theta)$

$\hat{r}=\frac{r}{|r|}$
$\hat{\phi}=(-\sin(\phi),\cos(\phi))$

$\hat{\theta}=\hat{\phi}\times\hat{r}=(\cos(\phi)\cos(\theta),\ \sin(\phi)\cos(\theta),\ -\sin(\theta))$

T: For two elementary regions in $\mathbb{R}^2$, D & D^*, & $T: D^*\longrightarrow D$ a C^1 bijection then

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(T(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv$$

Where $T(u,v)=(x(u,v),y(u,v))$ &

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\det\begin{bmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{bmatrix}\right|$$

Note that the jacobian of polar coordinates is always r, for cylindrical its ρ, for spherical its $r^2\sin(\theta)$.
We can generalise to n integrals by multiplying by an appropriate jacobian

T: For an integral with constant terminals & an integrand that factorises

ie $g(x,y,z) = g_1(x) g_2(y) g_3(z)$

we have that

$$\int_a^b\int_c^d\int_e^f g(x,y,z)\,dx\,dy\,dz = \int_e^f g_1(x)dx \int_c^d g_2(y)dy \int_a^b g_3(z)dz$$

The integral of a product is the product of the integral in this circumstance

Center of Mass:

A body with mass per unit volume given by $\mu(x,y,z)$ has **mass** given by

$$M = \iiint_D \mu(x,y,z)\,dx\,dy\,dz$$

& has **center of mass** given by...

$$x_{cm} = \frac{1}{M}\iiint_D x\,\mu(x,y,z)\,dx\,dy\,dz$$

replace with y, z for y_{cm}, z_{cm}.

For a 2D plate remove all the dz integrations.

D: I_n is the **moment of inertia** about the n axis. Measures the response of the body of spinning it about the n axis. (Higher $I_n \Rightarrow$ harder to spin)

If $\mu(x,y,z)$ is mass per unit volume of a body then

$$I_x = \iiint_D (y^2+z^2)\,\mu(x,y,z)\,dx\,dy\,dz.$$

For the other I_y & I_z have the two components not being looked at.

Path & Surface Integrals:

D: f a cts scalar function. c a C^1 path with $c(t) = (x(t), y(t), z(t))$. Then the **path integral** of f along c, over $t \in [a,b]$ is:

$$\int_c f\,ds = \int_a^b f(c(t))\,|c'(t)|\,dt$$

arc length element.

Note t must increase in the parametrisation. When $f=1$ the path integral gives the arclength of c.

D: F a cts vector field, c a C^1 path $c(t) = (x(t), y(t), z(t))$. The **line integral** of F over c over $t\in[a,b]$ is

$$\int_c F\cdot ds = \int_a^b F(c(t))\cdot c'(t)\,dt$$

A short hand for this is for $F = (u, v, w)$

$$\int_c F\cdot ds = \int_c u\,dx + v\,dy + w\,dz$$

This integral is also known as a work integral.

D: A **parametrised surface** is a function $\Phi : D \longrightarrow \mathbb{R}^3$, $D \subseteq \mathbb{R}^2$ a domain.

D: If Φ is a C^1 function then S is a **differentiable / C^1 surface.**

Tangents & normals to Surfaces:

Let S be a C^1 surface, consider curves on S $\quad C_1(v) = \Phi(u_0, v)$, $C_2(u) = \Phi(u, v_0)$ for a given (u_0, v_0)

D: T_v is the **tangent vector** to $C_1(v)$ at $\Phi(u_0,v_0)$

$$T_v = \frac{dc_1}{dv}\Big|_{v=v_0} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)\Big|_{(u_0,v_0)}$$

Similar for $T_u = \frac{dC_2}{du}\Big|_{u=u_0}$.

D: The **unit normal** to the surface at (x_0, y_0, z_0) is $n = T_u \times T_v$

OR $n' = -n$

T: $n(u_0, v_0) \neq 0 \Rightarrow$ Surface is smooth

D: For a surface S smooth at (x_0,y_0,z_0) The **cartesian equation of the tangent plane** at this point is

$$(x-x_0,\ y-y_0,\ z-z_0)\cdot \underline{n}(u_0,v_0) = 0$$

the normal to S.

D: S a smooth, except possibly at a finite # of points (EPFE), surface parametrised by $\phi: D \to S$, (input u & v). **The surface area** is

$$\iint_S dS = \iint_D |T_u \times T_v|\,du\,dv$$

T: f a cts function on a smooth EPFE surface S then

$$\iint_S f\,dS = \iint_D f(\phi(u,v))\,|T_u \times T_v|\,du\,dv$$

D: An **oriented surface** is a two sided surface.

T: Let F be a cts vector field on a smooth, EPFE, orientable, parametrised surface S

outward normal

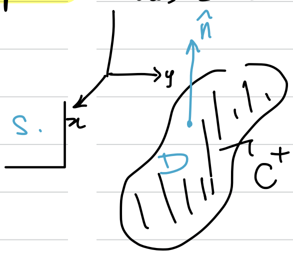
$$\Rightarrow \iint_S F\cdot d\underline{S} = \iint_S F\cdot \hat{n}\,dS$$

$$= \iint_D F\cdot (T_u \times T_v)\,du\,dv$$

with $\phi: D \to S$

$(u,v) \longmapsto (x(u,v), y(u,v), z(u,v))$

D: Positive orientation of a oriented closed curve in xy plane.



- $\hat{n}$ is normal to xy plane, in the direction of $\underline{k}$, it is related to C^+ by right hand rule (by thumb).
- If you walk along C^+ the region D will be on your left.
- restrict to simple closed curves

 simple  Non simple.

T: (Greens Theorem) in the plane:

$$\int_{C=\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

OR

$$\int_{\partial D} \underline{F}\cdot d\underline{s} = \iint_D (\nabla \times \underline{F})\cdot \underline{k}\,dx\,dy$$

Where:

- D is a region in xy plane bounded by simple closed curve $C = \partial D$ with positive orientation
- $\underline{F}$ is a C^1 vector field on D
  $F(x,y) = (P(x,y),\ Q(x,y))$
- D is composed of regions of both horizontal & vertical strips.

T: C a simple closed curve bounding a region D then

$$\text{Area of } D = \frac{1}{2}\int_C x\,dy - y\,dx$$

T: (Divergence in the Plane) Under the same conditions as Greens thm where $\hat{n}$ is the outward normal to ∂D in the xy plane. Then

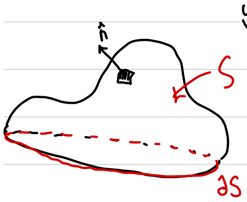
$$\int_{C=\partial D} F\cdot\hat{n}\,ds = \iint_D \nabla\cdot F\,dx\,dy$$

T: (Stokes Theorem)

- S an open oriented surface parametrised by $\Phi(u,v)$ a C^2 mapping
- ∂S oriented closed boundry of S
- F is a C^1 vector field on S
- S & ∂S are oriented such that $\hat{n}$ is the unit outward normal to S.
- Orientation of ∂S & $\hat{n}$ are related by right hand rule.

$$\iint_S (\nabla \times F)\cdot d\underline{S} = \int_{\partial S} F \cdot d\underline{s}$$

<u>Orientation:</u> Walk along boundary with $\hat{n}$ as your "up", you are moving in the positive direction if S is on your left.



T: Under conditions of Stokes theorem any two surfaces S_1 & S_2 with the same boundary C we have

$$\iint_{S_1} (\nabla \times F)\cdot d\underline{S} = \iint_{S_2} (\nabla \times F)\cdot d\underline{S} = \int_C F \cdot ds$$

T: A closed surface has

$$\iint_S (\nabla \times F)\cdot d\underline{S} = 0.$$

T: F C^1 vector field on $\mathbb{R}^2$ or $\mathbb{R}^3$.

- $\forall$ oriented, simple, closed curve C

$$\int_C F \cdot d\underline{s} = 0.$$

$\iff$ For any two oriented simple curves C_1, C_2 with same endpoints $\int_{C_1} F \cdot d\underline{s} = \int_{C_2} F \cdot d\underline{s}$

$\iff$ $F = \nabla \phi$ for some scalar func ϕ

$\iff$ $\nabla \times F = \underline{0}$. (irrotational)

D: An irrotational vector field is also called conservative.

T: (Gauss' Divergence Theorem)

- Ω a solid region in $\mathbb{R}^3$
- $\partial \Omega$ oriented closed surface
- F a C^1 vect field on Ω
- Orientation given by $\hat{n}$ unit outward normal.

$$\iiint_\Omega \nabla \cdot F \, d\underline{V} = \iint_{\partial \Omega} F \cdot d\underline{S}$$

T: Volume of $\Omega = \frac{1}{3}\iint_{\partial\Omega} (x,y,z)\cdot d\underline{S}$

Curve Linear Coordinates:

For each point in the cartesian plane (x,y,z) we associate a unique set of curvelinear coordinates (u_1,u_2,u_3) where:

$x=f_1(u_1,u_2,u_3)$, $y=f_2(u_1,u_2,u_3)$, $z=f_3(u_1,u_2,u_3)$

& $u_1=g_1(x,y,z)$, $u_2=g_2(x,y,z)$, $u_3=g_3(x,y,z)$

Let $r(x,y,z)=(x,y,z)$ be the position vector

Then $r(u_1,u_2,u_3)=f(u_1,u_2,u_3)=(f_1,f_2,f_3)(u_1,u_2,u_3)$

A tangent to (x,y,z) with u_2,u_3 const is $\frac{\partial r}{\partial u_1}$. This has length $\left|\frac{\partial r}{\partial u_1}\right|=h_1$

Thus a unit tangent in this direction is

$$e_1=\frac{1}{h_1}\frac{\partial r}{\partial u_1}$$

Similarly $\frac{\partial r}{\partial u_2}=\left|\frac{\partial r}{\partial u_2}\right|e_2=h_2e_2$

$\frac{\partial r}{\partial u_3}=h_3e_3$

D: We call h_i scale factors.

D: The coordinate system is orthogonal if $\forall i\neq j$ $e_i\cdot e_j=0$.

T: Let u_1,u_2,u_3 curve lin coords

Consider parametrised curve

$r(t)=r(u_1(t),u_2(t),u_3(t))$

$$\Rightarrow \frac{dr}{dt}=h_1e_1\frac{du_1}{dt}+h_2e_2\frac{du_2}{dt}+h_3e_3\frac{du_3}{dt}$$

Let $f:\mathbb{R}^3\to\mathbb{R}$ be a C^2 scalar function and $\boldsymbol{F}:\mathbb{R}^3\to\mathbb{R}^3$ be a C^1 vector field v

$$\boldsymbol{F}(u_1,u_2,u_3)=F_1(u_1,u_2,u_3)\mathbf{e}_1+F_2(u_1,u_2,u_3)\mathbf{e}_2+F_3(u_1,u_2,u_3)\mathbf{e}_3.$$

Then

1. $\nabla f=\frac{1}{h_1}\frac{\partial f}{\partial u_1}\mathbf{e}_1+\frac{1}{h_2}\frac{\partial f}{\partial u_2}\mathbf{e}_2+\frac{1}{h_3}\frac{\partial f}{\partial u_3}\mathbf{e}_3$
2. $\nabla\cdot\boldsymbol{F}=\frac{1}{h_1h_2h_3}\left[\frac{\partial(h_2h_3F_1)}{\partial u_1}+\frac{\partial(h_1h_3F_2)}{\partial u_2}+\frac{\partial(h_1h_2F_3)}{\partial u_3}\right]$
3. $\nabla\times\boldsymbol{F}=\frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\mathbf{e}_1 & h_2\mathbf{e}_2 & h_3\mathbf{e}_3\\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3}\\ h_1F_1 & h_2F_2 & h_3F_3\end{vmatrix}$
4. $\nabla^2 f=\frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial f}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial f}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right]$

T: Volume element

$$|\text{Jacobian}|=h_1h_2h_3\underbrace{|e_1\cdot(e_2\times e_3)|}_{=1 \text{ in orthonormal } e_1,e_2,e_3 \text{ case}}$$

T: Surface element

$$|T_u\times T_v|=h_uh_v\underbrace{|e_u\times e_v|}_{=1 \text{ in orthonormal } e_u,e_v \text{ case}}$$

Physical Interpretations:

velocity }
acceleration } of a path $\frac{dc}{dt}$, $\frac{d^2c}{dt^2}$

Curl $\longrightarrow$ $\nabla \times F$ F is fluid velocity, then curl is how quickly & in what direction the twig dropped in would rotate as it moves around.

Irrotational $\rightarrow$ $\nabla \times F = 0$.

Divergence $\longrightarrow$ F is fluid velocity then $\nabla \cdot F$ is net transport of fluid in/out of that point
$\nabla \cdot F > 0$ source, $\nabla \cdot F < 0$ sink

Incompressible $\longrightarrow$ $\nabla \cdot F = 0$.
Conservation $\longrightarrow$ $\nabla \cdot F = 0$.

Volume $\longrightarrow$ $\iint_D f(x,y)\,dA$ is volume under surface $f(x,y)$

Area $\longrightarrow$ $\iint_D 1\,dA$ is area of D. $\rightarrow$ $\iiint_D 1\,dV$ is volume of D.

Mass of a wire }
Charge of a wire } $\longrightarrow$ If f is
- mass / unit length
- charge / unit length

Then $\int_{C(t)} f\,ds$ is (line integral)

Work Done $\longrightarrow$ F a force field then $\int_C F \cdot d\underline{s}$ is work done by field in moving a particle along $\underline{C}$.

Mass of solid $\longrightarrow$ f is mass / unit volume $\iiint_D f\,dV$ is mass of D.

Flux $\longrightarrow$ $\iint_S F \cdot d\underline{S}$ surface integral

Flow lines $\longrightarrow$ A curve whose tangent coincides with the vector field

Maxwells Equations $\longrightarrow$

Example 3: Maxwell's Equations for Electromagnetic Fields.

Define the following quantities
- electric charge density – $\rho(\boldsymbol{r}, t)$
- electric current – $\boldsymbol{J}(\boldsymbol{r}, t)$
- vector field for magnetic force – $\boldsymbol{B}(\boldsymbol{r}, t)$
- vector field for electric force – $\boldsymbol{E}(\boldsymbol{r}, t)$
- permittivity of free space – ε_0
- permeability of free space – μ_0

In S.I. units, Maxwell's equations can be written as

(a) $\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$ (Faraday's Law)
(if $\boldsymbol{B}$ changes with time an electric field is produced)

(b) $\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0}$ (Gauss' Law)
(charges present make $\nabla \cdot \boldsymbol{E} \neq 0$)

(c) $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \varepsilon_0 \mu_0 \frac{\partial \boldsymbol{E}}{\partial t}$ (Ampere's Law)
(if $\nabla \times \boldsymbol{B} \neq \boldsymbol{0}$ then currents or $\boldsymbol{E}$ changes with time)

(d) $\nabla \cdot \boldsymbol{B} = 0$
($\boldsymbol{B}$ is always incompressible, no magnetic sources)

Consequences of Maxwell's equations are

1. If $\boldsymbol{B}$ is constant in time so $\boldsymbol{B}(\boldsymbol{r})$ only.

Center of mass →

- 2D plate with mass per unit area $\mu(x,y)$, the centre of mass is (x_{cm}, y_{cm}) where

$$x_{cm} = \frac{\iint_D x\,\mu(x,y)\,dx\,dy}{\text{mass}}$$

$$y_{cm} = \frac{\iint_D y\,\mu(x,y)\,dx\,dy}{\text{mass}}$$

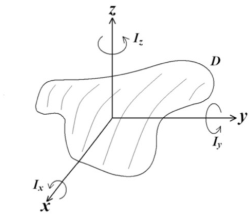
where

$$\text{mass of plate} = \iint_D \mu(x,y)\,dx\,dy.$$

Moment of inertia →

Moments of Inertia

- I_n is the moment of inertia of a solid body about the n axis.
- I_n measures a body's response to spinning it about the n axis



As I_n increases, it becomes harder to spin the body about the n axis.

- If $\mu(x,y,z)$ is the mass per unit volume of a solid body, then

$$I_x = \iiint_D (y^2+z^2)\mu(x,y,z)\,dx\,dy\,dz$$

$$I_y = \iiint_D (x^2+z^2)\mu(x,y,z)\,dx\,dy\,dz$$

$$I_z = \iiint_D (x^2+y^2)\mu(x,y,z)\,dx\,dy\,dz$$

Curvature → of a path is the angular rate of change of the direction of T per unit change in distance along the path.

Torsion → of a path is how fast the path twists out of the plane of T and N at c(t).